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Production and Inventory Rationing in a Make-to-Stock System With a Failure-Prone Machine and Lost Sales

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***Abstract*—We consider production and inventory rationing of a product to fulfill multiple demand classes in a make-to-stock production system with a failure-prone machine. Demand that cannot be satisfied immediately is lost and incurs a lost sales cost, which differs from class to class. We find that the optimal control policies under both the expected total discounted cost criterion and the average cost criterion have similar structural properties. Specifically, the optimal production policy is the base-stock policy and the optimal inventory allocation policy is the threshold control policy with machine-state-dependent threshold levels. Finally, we provide numerical examples to show the importance of taking machine failures into consideration and the effectiveness of the optimal control policy.**

***Index Terms*—Machine failures, multiclass demands, optimal control.**

I. Introduction

One of the challenges in managing production systems is to cope with machine unreliability. Production facilities are subject to unpredictable breakdowns due to age and usage. Machine failures render production processes uncertain and curtail production capacity. Considering that demands requiring the same product may have different values to different firms, require different service levels, or incur different penalties for delays, many firms adopt a demand differentiation strategy that segments demands into different classes and provides different services to different classes. Demand differentiation requires proper inventory allocation decisions to carry out. Under the situation where the on-hand inventory is capacitated and not enough to satisfy all the different demands, an inventory allocation decision needs to be made as to whether to satisfy the current requirement for the product from a certain demand class or reserve the inventory for the more important demands that will arrive in the future. The issue of inventory rationing in production systems with failure-prone machines exists in the real world. For example, many high-tech electronic manufacturing firms produce core components for different customers, e.g., Qualcomm, Intel, and AMD. But the lost sales costs associated with different customers vary significantly according to contractual agreements, which prompt manufacturers to implement appropriate inventory allocation policies to adjust product allocations among multiple demand classes.

There exists a large body of literature on the optimal control of production systems with failure-prone machines. [1], [2], [13], [17], and [18] consider the optimal control of production rates in manufacturing systems with deterministic demand, while [4] and [5] consider the case with stochastic demand. Another stream of literature studies combined preventive maintenance and production control. [12] extensively reviews the related literature on this topic and shows that the optimal control policy is dynamic and rather complicated. However, the existence of structural properties of the optimal control policy remains an open question. Almost all of the above literature is about the single demand class, which is technically equivalent to the situation where the lost sales cost of each demand class is equal in our model. Such demand classes can be aggregated as a single demand. Our study differs from such works in that we consider multiple demand classes where the lost sales cost varies from class to class. The optimal control policy characterizes the production decision, as well as the inventory allocation decision.

Inventory allocation has been widely studied in the literature. We mainly review the related literature on optimal control in queueing-like systems. The optimal rationing policy in a make-to-stock (MTS) system with exponential processing times and lost sales was studied in [8]. The author proves that the optimal production policy is the base-stock policy and the optimal allocation policy is the threshold control policy. [6] considers a similar model, but the authors assume that the advance demand information is imperfect. They show that the optimal control policy is state-dependent. Subsequently, [10] extends [8] and considers a system with Erlang distributed processing times. The optimal rationing policy is the so-called critical work storage level policy. Recently, [11] considers the batch ordering case. The authors prove that the optimal control policy has the threshold-type property. Different from the above lost sales models, some studies focus on the backorder model. [9] considers optimal control in a system with two demand classes and backlogs. The author derives that the optimal production policy is also the base-stock policy, but the optimal rationing policy has a monotone switching curve structure. [3] extends [9] to the multiple demand classes case. As an extension of [7], [10] studies an Erlang processing time model with backorders. The common solution approach is to formulate the system as a Markov decision process and establish monotonicity of the optimal control policy by the supermodularity of the objective function, see [14], [19], and [20]. These papers assume exponential or Erlang processing times, but our technical note considers a special case of general distributed processing times and explicitly considers the influence of machine failures on the optimal control policy.

The rest of the technical note is organized as follows: We introduce the basic model and characterize the optimal control policy under the expected total discounted cost criterion in Section II. We consider the average cost criterion in Section III. In Section IV, we provide some numerical examples to highlight the importance of taking machine failures into consideration. Finally, we conclude the technical note and suggest some future research topics in Section V.

II. The Expected Total Discounted Cost Criterion

Consider a production system with a single machine producing one type of product to stock. Inventory is used to satisfy demands from n different classes. Demand from class i, $i = 1, 2, \ldots, n$, arrives according to an independent Poisson process with a rate λ_i and requires one unit of the product. Let λ be the total demand rate, i.e., $\lambda = \sum_{i=1}^{n} \lambda_i$. Demand that cannot be satisfied immediately from stock is lost and incurs a lost sales cost c_i if it is from class i. Without loss of generality, we assume $c_1 > c_2 > \cdots > c_n$. Inventory is replenished by a failure-prone machine. The processing times are exponentially distributed with a rate μ. We assume that machine failures are time-dependent only, which implies that they are independent of whether or not the machine is operational, as studied by [4], [5]. The time duration between two successive breakdowns also follows an exponential distribution with a rate b. Once a machine failure happens, the machine is sent to repair immediately. We assume that the repair times are exponentially distributed with a rate r. The production interrupted by machine breakdowns is resumed once the machine is repaired. Due to the memoryless property of the exponential distribution, the remaining processing time is stochastically equivalent to initiating

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production from scratch. We make the exponential assumptions to ensure that the model is tractable.

A control policy includes the production decision and inventory allocation decision. An allocation decision must be made as to whether to satisfy the current requirement from a certain demand class or reserve the inventory for more important demands that will arrive in the future. The other decision is whether or not to produce when the machine is up. We investigate the optimal control policies under two different decision criteria: the expected total discounted cost and the average cost.

We use $(X(t), M(t))$ to denote the state of the system, where $X(t) \in Z^+$ denotes the on-hand inventory at time t and $M(t)$ denotes the machine state at time t. $M(t) = k$ denotes that the machine is in state k. When the machine is operational, $M(t) = 1$; otherwise $M(t) = 0$. The system has the state space Ω, $\Omega = \{Z^+ \times \{0,1\}$. Let $J^v(x,k) = J^v(X(0) = x, M(0) = k)$ be the expected total discounted cost over an infinite horizon under policy v with a starting state (x,k), i.e.,

$$J^v(x,k) = E\left[\int_0^{+\infty} e^{-\beta t}\left[h(X(t))dt + \sum_{i=1}^{n} c_i dN_i^v(t)\right]\right] \quad (1)$$

where β is the discount rate, $h(X(t))$ denotes the holding cost function when the on-hand inventory is $X(t)$, which is convex, nonnegative, and $h(\mathbf{0}) = 0$, and $N_i^v(t)$ is the number of class i demands that have not been satisfied from the on-hand inventory up to time t under policy v.

Policy v^* is said to be optimal if it minimizes the expected total discounted cost, i.e.,

$$J^{v^*}(x,k) = \min_v J^v(x,k). \quad (2)$$

To simplify notation, we drop the superscript u^* from $J^{u^*}(x,j)$ in the rest of the technical note. Following Lippman [15], we re-scale the time unit so that $\beta+\lambda+\mu+b+r = 1$. Then the optimal cost function $J^*(x,k)$ must satisfy the following optimality equations:

$$J^*(x,k) = TJ^*(x,k) \quad (3)$$

where T is an operator on the set of real-valued function $u(x,k)$ defined on the state space Ω

$$Tu(x,k) = h(x) + \sum_{i=1}^{n} \lambda_i \min[u(x,k)+c_i, H_i u(x,k)] + \mu\min[u(x,k), u(x+k,k)] + bu(x,0) + ru(x,1) \quad (4)$$

where $H_i, i = 1,2,\ldots,n$, is an operator defined as follows:

$$H_i u(x,k) = \begin{cases} u(x,k)+c_i, & x = 0; \\ u(x-1,k), & \text{otherwise.} \end{cases}$$

In (4), the first minimization operation is associated with the decision of whether or not to fill a newly arrived demand from class i and the second minimization operation is associated with the decision of whether or not to produce when the machine is up. Obviously, it is optimal to satisfy class i demand when $J^*(x,k) + c_i \geq J^*(x-1,k)$ and the on-hand inventory is positive; otherwise reject it. It is optimal to produce when $J^*(x,1) \geq J^*(x+1,1)$.

In order to characterize the structural properties of the optimal control policy, we introduce a set of functions with certain properties and prove that operator T preserves these properties.

Definition 1: $\mathcal{V}$ is a set of functions defined on Ω. If $u(x,k) \in \mathcal{V}$, then $u(x,k)$ satisfies the following properties:

$$C1: u(x+2,k) - u(x+1,k) \geq u(x+1,k) - u(x,k)$$
$$C2: u(x+1,1) - u(x,1) \geq u(x+1,0) - u(x,0)$$
$$C3: u(x+1,k) - u(x,k) \geq -c_1.$$

Lemma 1: If $u(x,j) \in \mathcal{V}$, then $Tu(x,j) \in \mathcal{V}$.

Proof: For any $u(x,k) \in \mathcal{V}$, we prove that $Tu(x,k)$ satisfies properties C1-C3. To facilitate analysis, let $m_i(x,k) = \min[u(x,k)+c_i, H_i u(x,k)]$, where $k = 0, 1$, and $n(x) = \min[u(x,1), u(x+1,1)]$. Define a difference operator D such that $Du(x,k) = u(x+1,k) - u(x,k)$, $Dm_i(x,k) = m_i(x+1,k) - m_i(x,k)$, and $Dn(x) = n(x+1) - n(x)$.

Proof of Property C1: First, we prove that $m_i(x,k)$ satisfies C1, i.e., $Dm_i(x+1,k) \geq Dm_i(x,k)$. We have two cases.

1) Case 1: $x > 0$. Consider $Du(x,k-1) \leq Du(x,k) \leq Du(x+1,k)$, which leads to the following three subcases.

1) Suppose $-c_i \leq Du(x-1,k) \leq Du(x+1,k)$, then we have $Dm_i(x+1,k) = u(x+1,k) - u(x,k) \geq u(x,k) - u(x-1,k) = Dm_i(x,k)$.
2) Suppose $Du(x-1,k) \leq -c_i \leq Du(x+1,k)$, then we have $Dm_i(x+1,k) \geq u(x+1,k) - [u(x+1,k)+c_i] = u(x,k) - [u(x,k)+c_i]m \geq Dm_i(x,k)$.
3) Suppose $Du(x-1,k) \leq Du(x+1,k) \leq -c_i$, then we have $Dm_i(x+1,k) = [u(x+2,k)+c_i] - [u(x+1,k)] \geq [u(x+1,k)+c_i] - [u(x,k)+c_i] = Dm_i(x,k)$.

2) Case 2: $x = 0$. We only have to prove that $Dm_i(1,k) \geq Dm_i(0,k)$. We distinguish the following two subcases.

1) Suppose $-c_i \leq Du(1,k)$, then $Dm_i(1,k) \geq u(1,k) - [u(1,k)+c_i] = u(0,k) - [u(0,k)+c_i] \geq Dm_i(0,k)$.
2) Suppose $Du(1,k) \leq -c_i$, then $Dm_i(1,k) = [u(2,k)+c_i] - [u(1,k)+c_i] \geq [u(1,k)+c_i] - [u(0,k)+c_i] \geq Dm_i(0,k)$.

Since $n(x)$ is equal to $m_i(x)$ with $c_i = 0$, $n(x)$ also satisfies C1. Furthermore, the other terms in $Tu(x,k)$ are convex functions, which means that $Tu(x,k)$ is convex in x for any k, $k = 0, 1$.

Proof of Property C2: First, we prove that $m_i(x,k)$ satisfies C2, i.e., $Dm_i(x,1) \geq Dm_i(x,0)$. If $x > 0$, consider four cases.

1) Suppose $-c_i \leq Du(x-1,0) \leq Du(x,1)$, then $Dm_i(x,1) \geq u(x,1) - u(x-1,1) \geq u(x,0) - u(x-1,0) = Dm_i(x,0)$.
2) Suppose $Du(x-1,0) \leq -c_i \leq Du(x,1)$, then $Dm_i(x,1) \geq u(x,1) - [u(x,1)+c_i] = u(x,0) - [u(x,0)+c_i] = Dm_i(x,0)$.
3) Suppose $Du(x-1,0) \leq Du(x,1) \leq -c_i$, then $Dm_i(x,1) = [u(x+1,1)+c_i] - [u(x,1)+c_i] \geq [u(x+1,0)+c_i] - [u(x,0)+c_i] = Dm_i(x,0)$.

Similarly, we can show that C2 holds when $x = 0$. So $m_i(x,j)$ satisfies C2. Next we show that $Dn(x) \geq u(x+1,0) - u(x,0)$. When $x > 0$, there are two cases.

1) Suppose $Du(x+1,1) \leq 0$, then $Dn(x) = u(x+2,1) - u(x+1,1) \geq u(x+1,1) - u(x,1) \geq u(x+1,0) - u(x,0)$.
2) Suppose $Du(x+1,1) > 0$, then $Dn(x) \geq u(x+1,1) - u(x,1) \geq u(x+1,0) - u(x,0)$. So T preservers property C2.

Proof of Property C3: First, we prove that $m_i(x,k)$ satisfies C3, i.e., $Dm_i(x,k) \geq -c_1$. If $x > 0$, we have the following two subcases.

1) Suppose $-c_i \leq Du(x,k)$, then we have $Dm_i(x,k) \geq u(x,k) - u(x-1,k) \geq -c_1$.
2) Suppose $Du(x,k) \leq -c_i$, then we have $Dm_i(x,k) \geq [u(x+1,k)+c_i] - [u(x-1,k)+c_i] \geq -c_1$.

In a similar manner, we find that $m_i(x,k)$ satisfies C3 when $x = 0$. $n(x)$ is a special case of $m_i(x,k)$ with $c_i = 0$, which implies that $n(x)$ satisfies C3. Then we have $Tu(x+1,1) - Tu(x,1) = h(x+1) - h(x) + \sum_{i=1}^{i=n} \lambda_i Dm_i(x,1) + \mu Dn(x) + bDu(x,0) + rDu(x,1) \geq -c_1\left[\sum_{i=1}^{i=n} \lambda_i + \mu + b + r\right] > -c_1$.

Following a similar process, we obtain that $Tu(x+1,0) - Tu(x,0) \geq -c_1$. So T preserves property C3. Lemma 1 is proved.

Proposition 1: The optimal cost function $J^*(x,j) \in \mathcal{V}$. The optimal control policy is a base-stock/threshold policy. Furthermore

$$P1: 0 = R_1(k) \leq R_2(k) \leq \cdots \leq R_n(k)$$
$$P2: R_i(1) \leq R_i(0), \text{ for } i = 1,2,\ldots,n$$

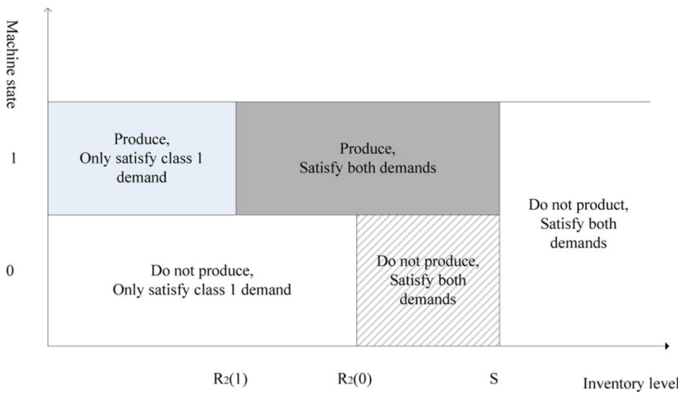

Fig. 1. Structure of the optimal policy.

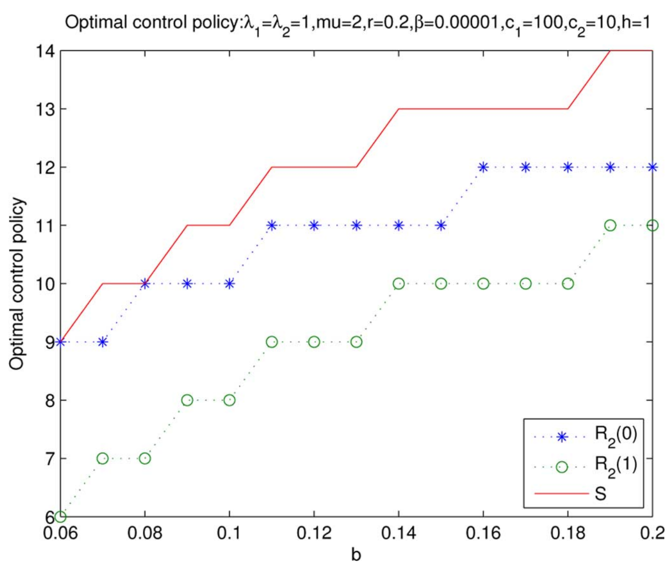

Fig. 2. Optimal policy versus b.

where $R_i(k)$, $i = 1, 2, \ldots, n$, $k = 0,\ 1$, is the threshold level for demand i when the machine is in state k.

Proof: Lemma 1 shows that operator T preserves properties C1, C2, and C3, which implies that $J^*(x, j) \in \mathcal{V}$. It is optimal to produce in state $(x, 1)$ if $J^*(x+1, 1) - J^*(x, 1) \le 0$ and to satisfy demand from class i in state (x, k) if $J^*(x, k) - J^*(x-1, k) \ge -c_i$. Since $J^*(x+1, k) - J^*(x, k)$ is increasing in x due to convexity, a base-stock/threshold policy is optimal.

As for properties P1, $R_{i-1}(k) \le R_i(k)$ is due to convexity; C3 implies $R_1(k) = 0$. We have $-c_i \le J(R_i(0)+1, 0) - J(R_i(0), 0) \le J(R_i(0)+1, 1) - J(R_i(0), 1)$ due to supermodularity, which implies $R_i(1) \le R_i(0)$. Proposition 1 is proved.

The optimal base-stock level satisfies $S = \min_{x \ge 0}\{x : J^*(x+1, 1) - J^*(x, 1) \ge 0\}$, while the optimal threshold level is uniquely defined and satisfies $R_i(k) = \min_{x \ge 0}\{x : J^*(x+1, k) - J^*(x, k) \ge -c_i\}$. We also have $R_n(1) \le S$ due to the definition of S, i.e., $J^*(S+1, 1) - J^*(S, 1) \ge 0$. The optimal policy works in the following manner: Produce only when $X(t) < S$ and the machine is up; satisfy demand from class i only when $X(t) > R_i(k)$, otherwise do not fulfill it. The inventory allocation decision is machine-state-dependent. The threshold level associated with a certain demand class when the machine is down is larger than that when the machine is up. This is very intuitive because production is interrupted by machine breakdowns and we should reserve more inventory to fulfill demands with higher lost sales costs. Therefore if it is optimal to satisfy demand from class i when the machine is up, but once a machine failure happens, continuing to satisfy that demand from the on-hand inventory may not be optimal. But if it is optimal to satisfy demand from a certain class when the machine is down, it remains optimal to satisfy that demand when the machine is up. Proposition 1 also states that we should never reject demand from class 1. The structure of the optimal control policy is shown in Fig. 1, in which we specify the optimal actions in each region.

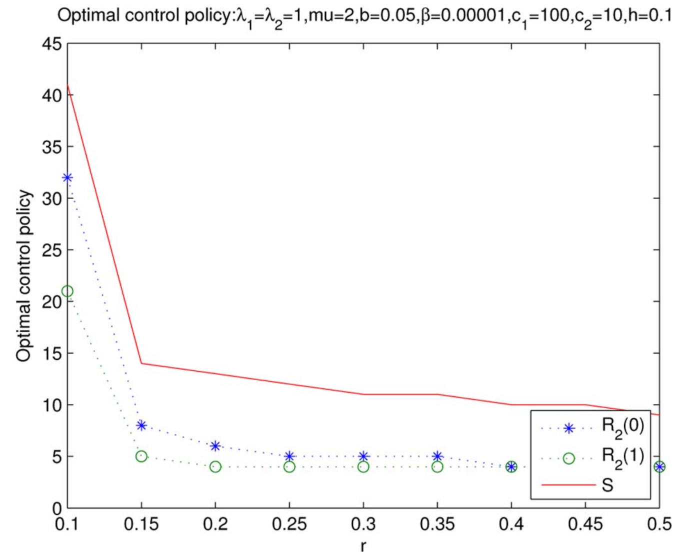

Fig. 3. Optimal policy versus r.

III. Average Cost Criterion

In this section we investigate the structural properties of the optimal control policy under the average cost criterion. Let $g^{\pi^*}(x, k)$ denote the optimal average cost function with starting state (x, k) and π^* be the optimal control policy, i.e.

$$g^{\pi^*}(x, k) = \min_{\pi} \lim_{\tau \to +\infty} \frac{1}{\tau} E\left[\int_0^{\tau} \left[h(X(t))dt + \sum_{i=1}^{n} c_i dN_i^{\pi}(t)\right]\right]. \tag{5}$$

Consider a stationary policy π. Under such a policy, the system operates in the following manner: Production is controlled by the base-stock policy when the machine is up. Inventory is allocated to demands in the FCFS order regardless of their classes as long as the on-hand inventory is positive. The induced Markov chain is positive irreducible and the model is unichain, so there exists a finite constant average cost g^{π}. On the other hand, $h(x)$ is a convex function of x and positive, so the number of x that satisfies $h(x) < g$ is positive and finite. Therefore, there exist an optimal constant average cost g^* independent of the initial state and a bias $f(x, k)$ that satisfies the following equation:

$$f(x, k) = T' f(x, k) = \frac{1}{\nu}[T f(x, k) - g^*] \tag{6}$$

where $\nu = \lambda + \mu + b + r$; see [16]. The bias $f(x, k)$ denotes the expected total difference between the optimal constant average cost and the stationary cost. Then the optimal control policy is determined through the function $f(x, k)$. The following lemma establishes the properties of the optimal policy.

Lemma 2: If $f \in \mathcal{V}$, then $T' f(x, k) \in \mathcal{V}$.

Proof: A linear transformation of T and T' preserves properties C1–C3, confirming that the optimal control policy with respect to the average cost criterion possesses the same properties.

TABLE I
OPTIMAL CONTROL POLICY VS. THE OPTIMAL POLICY FOR THE FAILURE-FREE SYSTEM UNDER AVERAGE COST CRITERION

No.	λ_1	λ_2	μ	c_1	c_2	h	b	r	$(R1, R0, S)$	J^*	(R, S)	J^N	(%)
1	1	0.8	2.0	100	10	0.1	0.05	0.2	(9,17,23)	2.751	(6,17)	3.438	24.946
2	-	0.9	-	-	-	-	-	-	(10,17,23)	2.871	(6,18)	3.600	25.421
3	-	1.0	-	-	-	-	-	-	(10,17,25)	2.998	(6,20)	3.759	25.360
4	-	1.1	-	-	-	-	-	-	(11,18,27)	3.134	(7,22)	3.725	18.854
5	-	1.2	-	-	-	-	-	-	(11,18,28)	3.274	(7,24)	3.899	19.078
6	-	1.0	2.1	-	-	-	-	-	(9,17,23)	2.834	(6,18)	3.499	23.458
7	-	-	2.2	-	-	-	-	-	(8,16,22)	2.697	(5,16)	3.512	30.238
8	-	-	2.3	-	-	-	-	-	(7,15,20)	2.582	(5,15)	3.328	28.890
9	-	-	2.4	-	-	-	-	-	(6,15,19)	2.485	(4,13)	3.475	39.821
10	-	-	2.5	-	-	-	-	-	(6,14,18)	2.402	(4,12)	3.369	40.220
11	-	-	2.0	-	-	-	0.01	-	(5,14,15)	1.800	(4,14)	2.193	21.840
12	-	-	-	-	-	-	0.02	-	(6,15,18)	2.175	(5,15)	2.570	18.185
13	-	-	-	-	-	-	0.03	-	(7,16,20)	2.489	(5,17)	3.072	23.383
14	-	-	-	-	-	-	0.04	-	(9,17,23)	2.760	(6,18)	3.343	21.126
15	-	-	-	-	-	-	0.05	0.1	(21,32,41)	4.973	(9,30)	6.227	25.217
16	-	-	-	-	-	-	-	0.2	(10,17,25)	2.998	(6,20)	3.759	25.360
17	-	-	-	-	-	-	-	0.3	(7,13,20)	2.300	(5,17)	2.875	24.992
18	-	-	-	-	-	-	-	0.4	(6,10,17)	1.958	(5,16)	2.314	18.182
19	-	-	-	200	-	-	-	0.2	(15,23,30)	3.510	(8,22)	5.027	43.216
20	-	-	-	300	-	-	-	-	(18,26,33)	2.502	(9,23)	6.114	60.283
21	-	-	-	400	-	-	-	-	(21,28,36)	3.815	(9,23)	7.529	86.708
22	-	-	-	500	-	-	-	-	(22,30,37)	4.032	(10,24)	8.123	93.331
23	1.0	0.8	2.0	300	10	0.1	0.05	0.2	(17,25,31)	3.564	(8,19)	6.209	74.206
24	-	0.9	-	-	-	-	-	-	(18,25,32)	3.685	(9,21)	5.916	60.555
25	-	1.0	-	-	-	-	-	-	(18,26,33)	3.815	(9,23)	6.111	60.215
26	-	1.1	-	-	-	-	-	-	(19,26,35)	3.950	(9,25)	6.350	60.762
27	-	1.2	-	-	-	-	-	-	(19,26,36)	4.090	(9,26)	6.609	61.568
28	-	1.0	2.1	-	-	-	-	-	(17,25,31)	3.623	(8,20)	6.151	69.778
29	-	-	2.2	-	-	-	-	-	(16,24,29)	3.465	(7,18)	6.290	81.552
30	-	-	2.3	-	-	-	-	-	(15,23,28)	3.332	(6,17)	6.444	93.397
31	-	-	2.4	-	-	-	-	-	(14,22,26)	3.220	(6,15)	6.323	96.348
32	-	-	2.5	-	-	-	-	-	(13,22,25)	3.125	(6,14)	6.145	96.630
33	-	-	2.0	-	-	-	0.01	-	(8,20,20)	2.311	(6,15)	3.124	35.156
34	-	-	-	-	-	-	0.02	-	(11,21,24)	2.824	(7,17)	4.041	43.097
35	-	-	-	-	-	-	0.03	-	(14,23,28)	3.203	(7,19)	5.128	60.125
36	-	-	-	-	-	-	0.04	-	(16,24,31)	3.524	(8,21)	5.673	60.998
37	-	-	-	-	-	-	0.05	0.1	(40,51,60)	6.860	(13,34)	12.081	76.113
38	-	-	-	-	-	-	-	0.2	(18,26,33)	3.815	(9,23)	6.114	60.283
39	-	-	-	-	-	-	-	0.3	(12,18,25)	2.831	(8,19)	4.000	41.286
40	-	-	-	-	-	-	-	0.4	(10,14,21)	2.356	(7,18)	3.176	34.799
41	-	-	-	-	-	-	-	0.5	(8,12,19)	2.082	(7,17)	2.560	22.956

From Lemma 2, we see that the optimal control policy under the average cost criterion retains the same structural properties as those under the expected total discounted cost criterion.

IV. NUMERICAL EXAMPLES

In this section, we present some numerical examples to show how the optimal control policy responds to changes in system parameters and investigate the effectiveness of the optimal control policy by comparing it with the policy that is optimal for the failure-free system. The comparisons also highlight the importance of taking machine failures into consideration.

Figs. 2 and 3 indicate that the base-stock level and threshold levels have the relationships as stated in Proposition 1. Specifically, the base-stock level and threshold levels increase with the failure rate (Fig. 2), and decrease with the repair rate (Fig. 3). This is intuitive because the system, to cope with machine breakdowns, should hold more inventory to satisfy the demands from higher classes. In addition, as the repair rate increases, the influence of machine breakdowns on the system becomes less, so further increases in the base-stock level and threshold levels are not necessary.

In order to show the importance of taking machine failures into account and the benefit of adopting the optimal three-parameter threshold policy, we compare the following two policies: one is the optimal control policy and the other is the system operating under the policy that is optimal for the modified system without failures. We assume that the processing times of the modified system follow an exponential distribution with a rate $\mu r/(r+b)$, which is equal to the expected total processing time of the original system, i.e., the sum of the actual processing time and total repair time of the original system. The optimal cost function and the optimal control policy under both criterions can be computed by the value iteration algorithm. The reader is referred to [16] for details on the value iteration algorithm. In our experiments, we adopted the average cost criterion to evaluate the two systems. In order to apply the value iteration algorithm, we truncated the infinite countable state space to $[0, \bar{S}]$, where $\bar{S}$ is sufficiently large, to ensure that the optimal cost function is not sensitive to the truncated state space. We measure the effectiveness of the optimal control policy by the suboptimality $g^N - g^*/g^* * 100\%$, where g^N denotes the average cost under the modified system. The suboptimality also measures the significance of machine failures.

Table I presents the computational results. In order to examine the benefit of adopting the more complicated optimal three-parameter control policy over the two-parameter threshold policy, we chose the parameters as follows: 1) machine availability $r/(r+b)$ ranged from

0.67 to 0.95; (2) the system load $(\lambda_1 + \lambda_2)(b + r)/\mu r$ ranged from 0.75 to 1.38. From Table I, we see that the suboptimality ranges from $18.182\% - 96.630\%$, which indicates that machine failures significantly affect system performance. We summarize the main observations as follows.

- The optimal control policy performs better, especially when the differences between the lost sales costs are large, which implies the importance of taking machine failures into consideration.
- $R_i(k)$ and S all increase with b, c_1, and λ_2, and decrease with r. We also see that $R_i(k)$ decreases with c_2, while S increases with c_2

V. Conclusion

The authors extend [8] to the case with a failure-prone machine and characterize the structural properties of the optimal control policy under two different decision criteria. We find that the optimal control retains the threshold-type property, but is different in that the optimal threshold levels decrease with machine states. We highlight the importance of taking machine failures into account and make some interesting observations from the numerical examples.

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Denominator Assignment, Invariants and Canonical Forms Under Dynamic Feedback Compensation in Linear Multivariable Systems

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***Abstract*—A result orinally reported by Hammer [6] for linear time invariant (LTI) single input-single output systems and concerning an invariant and a canonical form of the transfer function matrix of the closed loop system under dynamic feedback compensation is generalized for LTI multivariable systems. Based on this result, we characterize the class of transfer function matrices that are obtainable from an open loop transfer function matrix via the use of *proper* dynamic feedback compensators and show that if the closed loop transfer function matrix $P_c(s)$ has a desired denominator polynomial matrix which satisfies a certain sufficient condition, then there exists a proper compensator giving rise to an internally stable closed loop system, whose transfer function matrix is $P_c(s)$.**

***Index Terms*—Decoupling, denominator assignment, Euclidean algorithm, proper feedback compensators.**

I. Introduction

Let Σ be a linear, time invariant (LTI), stabilizable multivariable system characterized by a *strictly proper* transfer function matrix $P(s)$ and consider the transfer function matrix $P_c(s)$ of the closed loop feedback system Σ_c in Fig. 1 where $C(s)$ is the transfer function matrix of a *proper* dynamic compensator. In this technical note, using mainly the Euclidean division for polynomial matrices [1], [7], [8], we first generalize to the multivariable case a result (originally reported in [6] for single input-single output systems) that concerns an invariant and a canonical form of the transfer function matrix $P_c(s)$ of the closed loop system Σ_c obtained from $P(s)$ via feedback through a proper compensator $C(s)$. This result leads to the characterization of the class $[P(s)]_{\mathcal{R}}$ of closed loop transfer function matrices $P_c(s)$ that are obtainable from $P(s)$ via the use of *proper* dynamic feedback compensators $C(s)$. We next determine the class $\mathcal{P}$ of open loop transfer function matrices $P(s)$ that give rise to an internally stable closed loop system Σ_c

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